



Center for Foundations of Intelligent Systems

Technical Report
98-13

Topologies, Continuity and
Bisimulations

J. M. DAVOREN

October 1998

19990621 044

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625 Rhodes Hall, Ithaca, NY 14853 (607) 255-8005

REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

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1. AGENCY USE ONLY (Leave Blank)	2. REPORT DATE 1 March 1999	3. REPORT TYPE AND DATES COVERED TECHNICAL	
4. TITLE AND SUBTITLE TOPOLOGIES, CONTINUITY AND BISIMULATIONS		5. FUNDING NUMBERS DAAH04-96-1-0341	
6. AUTHOR(S) J. M. DAVOREN			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Regents of the University of California c/o Sponsored Projects Office 336 Sproul Hall Berkeley, CA 94720-5940		8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211		10. SPONSORING / MONITORING AGENCY REPORT NUMBER <i>ARO 35873.98-MA-MUR</i>	
11. SUPPLEMENTARY NOTES The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.			
12 a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.		12 b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) The notion of a <i>bisimulation relation</i> is of basic importance in many areas of computation theory and logic. Of late, it has come to take a particular significance in work on the formal analysis and verification of <i>hybrid control systems</i> , based on the modal μ -calculus. Our purpose here is to give an analysis of the concept, starting with the observation that the zig-zag conditions are suggestive of some form of continuity. We give a topological characterization of bisimilarity for preorders, and then use the topology as a route to examining the <i>algebraic semantics</i> for the μ -calculus, developed in recent work of Kwiatkowska <i>et al.</i> , and its relation to the standard set-theoretic semantics.			
14. SUBJECT TERMS Bisimulations, fixed points, modal and temporal logic, formal verification, continuity of relations		15. NUMBER OF PAGES 16	
		16. PRICE CODE	
17. SECURITY CLASSIFICATION OR REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION ON THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

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Topologies, Continuity and Bisimulations

J. M. Davoren*

Center for Foundations of Intelligent Systems
626 Rhodes Hall, Cornell University
Ithaca, NY 14853, U.S.A.
email: davoren@hybrid.cornell.edu.

25 October 1998

Abstract. —The notion of a *bisimulation relation* is of basic importance in many areas of computation theory and logic. Of late, it has come to take a particular significance in work on the formal analysis and verification of *hybrid control systems*, based on the modal μ -calculus. Our purpose here is to give an analysis of the concept, starting with the observation that the zig-zag conditions are suggestive of some form of continuity. We give a topological characterization of bisimilarity for preorders, and then use the topology as a route to examining the *algebraic semantics* for the μ -calculus, developed in recent work of Kwiatkowska *et al*, and its relation to the standard set-theoretic semantics.

*Research supported by the ARO under the MURI program “Integrated Approach to Intelligent Systems”, grant no. DAA H04-96-1-0341.

1 Introduction

The notion of a *bisimulation relation* is of basic importance in many areas of computation theory and logic. In the propositional modal μ -calculus, if states x and y of *labeled transition system* (LTS) models \mathfrak{M} and \mathfrak{N} are bisimilar, then in their respective models, x and y satisfy all the same sentences of the language of L_μ . The corresponding properties of bisimulation-invariance for other formalisms are also well-studied: e.g. finitary and infinitary polymodal or temporal logics, and fragments of first-order, infinitary, and monadic second-order logics (see [8]).

The background motivation for this paper is the use of bisimulations in recent work on the formal analysis and verification of *hybrid control systems* (see [7], [4], and references therein). In that work, the computational model is a structure called a *hybrid automaton*, which is an enrichment of a *(real) timed automaton*. Temporal logic or μ -calculus specifications for such systems are interpreted with respect to LTS models \mathfrak{M} over states spaces $X \subseteq Q \times \mathbb{R}^n$, where Q is a finite set of control modes, and the transition relations are of two kinds: *continuous evolution* for some duration of time according to the differential equations modeling a given control mode, and relations modeling the effects of *discrete jumps* between control modes, which may be controlled or autonomous. The propositional constants denote sets of initial states, guard conditions on the jumps transitions, and goal or desired invariant regions of the state space. The systems of interest are those in which all the components of the associated LTS model \mathfrak{M} – the state space, transition relations and constant sets – are all *first-order definable* in some structure $\bar{\mathbb{R}} = (\mathbb{R}; <, +, \cdot, 0, 1, \dots)$ over the reals. For definiteness, take $\bar{\mathbb{R}}$ to be the real-closed field (which admits elimination of quantifiers), or more generally, take $\bar{\mathbb{R}}$ to be an *o-minimal structure* over \mathbb{R} (see [10], [6]).

Symbolic model checking tools for hybrid and real-time systems such as HYTECH [7] and KRONOS attempt to compute, for propositional μ -calculus sentences φ , the value of the denotation $\|\varphi\|^\mathfrak{M}$ as a *first-order formula* in the language $\mathcal{L}(\bar{\mathbb{R}})$, building up from the explicit first-order definitions of the components of \mathfrak{M} . For purely modal sentences, such a translation is straightforward. But for μ -sentences, to have a guarantee that the denotation $\|\mu Z. \varphi\|^\mathfrak{M}$ is a *finite* union of approximations, one needs to ensure that the LTS model \mathfrak{M} has a *bisimulation equivalence* \approx of *finite index*. If such is the case, the quotient LTS \mathfrak{M}_\approx is a finite truth-preserving simulacrum, and finite automata representation, of the original system. The construction of a finite

bisimulation quotient is the essential ingredient of results on the decidability of the reachability problem for a variety of first-order syntactic classes of hybrid and timed automata (see [10], and references therein).

Given their practical and theoretical significance, it behooves us to have a closer examination of the concepts involved.

The remainder of this paper is organized as follows. Section 2 is a review of the modal μ -calculus and bisimulation relations. In section 3, we give a topological characterization of bisimilarity for preorders. Section 4 makes the connection with algebraic semantics, and section 5 is a brief discussion of related research.

2 Background

Call a pair (Φ, Σ) consisting of a set Φ of propositional constants and a set Σ of transition (action) labels a *modal signature*, and let PVar be a fixed set of propositional variables. The set of formulas $\mathcal{F}_\mu(\Phi, \Sigma)$ in the signature (Φ, Σ) of the propositional modal μ -calculus is generated by the grammar:

$$\varphi ::= p \mid Z \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \langle a \rangle \varphi \mid \mu Z. \varphi$$

for $p \in \Phi$, $Z \in \text{PVar}$, and $a \in \Sigma$, with the proviso that in $\mu Z. \varphi$, the variable Z occur *positively*, i.e. each occurrence of Z in φ is within the scope of an even number of negations. Let $\mathcal{S}_\mu(\Phi, \Sigma)$ denote the set of all *sentences* of $\mathcal{F}_\mu(\Phi, \Sigma)$; i.e. those without any free variables. The logical constants tt and ff , other propositional connectives, dual modal operators $[a]$ and greatest fixed-point quantifier are defined in the standard way. For formulas $\varphi, \psi \in \mathcal{F}_\mu(\Phi, \Sigma)$, let $\varphi[Z := \psi]$ denote the result substituting ψ for all free occurrences of Z . By renaming bound variables in φ if necessary, we can assume such substitutions do not result in the unintended capture of free variables.

A *labeled transition system* (LTS) of signature (Φ, Σ) is a structure:

$$\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p \in \Phi})$$

where $X \neq \emptyset$ is the state space (of arbitrary cardinality); for each transition label $a \in \Sigma$, $a^{\mathfrak{M}} \subseteq X \times X$ is a binary relation on X ; and for each atomic proposition (observation or event label) $p \in \Phi$, $\|p\|^{\mathfrak{M}} \subseteq X$ is a unary relation on X .

A binary relation or set-valued map $r : X \rightsquigarrow Y$ (i.e. $r \subseteq X \times Y$) determines two *pre-image operators*: the *lower* or *existential* pre-image operator $\sigma(r) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is given by

$$\begin{aligned}\sigma(r)(B) &\doteq \{x \in X \mid (\exists y \in Y)[(x, y) \in r \wedge y \in B]\} \\ &= \{x \in X \mid r(x) \cap B \neq \emptyset\}\end{aligned}$$

for $B \subseteq Y$, while the *upper* or *universal* pre-image operator $\tau(r) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is the dual under set-theoretic complement:

$$\tau(r)(B) \doteq -\sigma(r)(-B) = \{x \in X \mid r(x) \subseteq B\}$$

The *direct image* operator mapping a set $A \subseteq X$ to its image in Y under r is just $\sigma(\check{r}) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, where $\check{r} : Y \rightsquigarrow X$ is the converse relation; in particular, the image of a point $r(x) = \sigma(\check{r})(\{x\})$.

In the standard set-theoretic semantics for the μ -calculus ([9], [14], [13]) over LTS models \mathfrak{M} , propositional variables range over the full power-set algebra $\mathcal{P}(X)$ of the state space. In the more general *algebraic semantics* of Kwiatkowska and colleagues in [1] and [2], formulas are interpreted with respect to *modal frames* $(\mathfrak{M}, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *modal algebra*: a Boolean algebra under the finitary set-theoretic operations, which contains each of the observation sets $\|p\|^\mathfrak{M}$ and is closed under each of the pre-image operators $\sigma(a^\mathfrak{M})$. We give the standard set-theoretic semantics here, and return to the algebraic semantics, and the relationship between the two, in Section 4.

Definition 2.1 *Given an LTS model $\mathfrak{M} = (X, \{a^\mathfrak{M}\}_{a \in \Sigma}, \{\|p\|^\mathfrak{M}\}_{p \in \Phi})$ of modal signature (Φ, Σ) , a (propositional, or second-order) variable assignment in \mathfrak{M} is any map $\xi : \text{PVar} \rightarrow \mathcal{P}(X)$. Each such assignment ξ uniquely extends to a denotation map $\|\cdot\|_\xi^\mathfrak{M} : \mathcal{F}_\mu(\Phi, \Sigma) \rightarrow \mathcal{P}(X)$ inductively defined as follows:*

$$\begin{aligned}\|p\|_\xi^\mathfrak{M} &\doteq \|p\|^\mathfrak{M} && \text{for } p \in \Phi \\ \|Z\|_\xi^\mathfrak{M} &\doteq \xi(Z) && \text{for } Z \in \text{PVar} \\ \|\neg\varphi\|_\xi^\mathfrak{M} &\doteq X - \|\varphi\|_\xi^\mathfrak{M} \\ \|\varphi_1 \vee \varphi_2\|_\xi^\mathfrak{M} &\doteq \|\varphi_1\|_\xi^\mathfrak{M} \cup \|\varphi_2\|_\xi^\mathfrak{M} \\ \|\langle a \rangle \varphi\|_\xi^\mathfrak{M} &\doteq \sigma(a^\mathfrak{M}) (\|\varphi\|_\xi^\mathfrak{M}) && \text{for } a \in \Sigma \\ \|\mu Z. \varphi\|_\xi^\mathfrak{M} &\doteq \bigcap \{A \in \mathcal{P}(X) \mid \|\varphi\|_{\xi(A/Z)}^\mathfrak{M} \subseteq A\}\end{aligned}$$

where for $A \in \mathcal{P}(X)$, the variant assignment $\xi(A/Z) : \text{PVar} \rightarrow \mathcal{P}(X)$ is given by: $\xi(A/Z)(W) = \xi(W)$ if $W \neq Z$, and $\xi(A/Z)(W) = A$ if $W = Z$.

For formulas $\varphi \in \mathcal{F}_\mu(\Phi, \Sigma)$ and assignments $\xi : \text{PVar} \rightarrow \mathcal{P}(X)$ in \mathfrak{M} , we say:

- φ is satisfied at state x in (\mathfrak{M}, ξ) , written $\mathfrak{M}, \xi, x \models \varphi$, iff $x \in \|\varphi\|_\xi^\mathfrak{M}$;
- φ is true in (\mathfrak{M}, ξ) , written $\mathfrak{M}, \xi \models \varphi$, iff $\|\varphi\|_\xi^\mathfrak{M} = X$; i.e. φ is satisfied at all states x in (\mathfrak{M}, ξ) ; and
- φ is true in \mathfrak{M} , written $\mathfrak{M} \models \varphi$, iff φ is true in (\mathfrak{M}, ξ) for all assignments ξ in \mathfrak{M} .

For sentences $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, the denotation $\|\varphi\|_\xi^\mathfrak{M}$ is independent of the variable assignment ξ , and is written $\|\varphi\|^\mathfrak{M}$. So $\mathfrak{M} \models \varphi$ iff $\mathfrak{M}, \xi \models \varphi$ for any assignment ξ .

The syntactic restriction on formulas $\mu Z. \varphi$ ensures that the operator $\varphi_{\xi, Z}^\mathfrak{M} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ given by $(\varphi_{\xi, Z}^\mathfrak{M})(A) \doteq \|\varphi\|_{\xi(A/Z)}^\mathfrak{M}$ is \subseteq -monotone. In the definition above, $\|\mu Z. \varphi\|_\xi^\mathfrak{M}$ is defined to be the least pre-fixed-point of $\varphi_{\xi, Z}^\mathfrak{M}$. By the Tarski-Knaster fixed-point theorem for monotone maps on complete lattices, least pre-fixed-points are the same as least fixed-points; thus the inclusion can be replaced with equality. The completeness of $\mathcal{P}(X)$ as a lattice ensures (by the Hitchcock-Park fixed-point theorem) that the set $\|\mu Z. \varphi\|_\xi^\mathfrak{M}$ may also be characterized as a transfinite union of an \subseteq -chain of approximations $\|\mu Z. \varphi\|_{\xi, \alpha}^\mathfrak{M}$ for ordinals α (of cardinality less than or equal to that of X), beginning with the empty set, applying the $\varphi_{\xi, Z}^\mathfrak{M}$ operator at successor ordinals and taking unions at limits.

Definition 2.2 Given two LTS's \mathfrak{M} and \mathfrak{N} , with state spaces X and Y respectively, a relation $\preccurlyeq : X \sim Y$ is called a bisimulation or zig-zag between \mathfrak{M} and \mathfrak{N} iff for $x, x' \in X$, $y, y' \in Y$ and each $a \in \Sigma$ and $p \in \Phi$,

- Zig_a:** $x \preccurlyeq y$ and $x \xrightarrow{a^\mathfrak{M}} x' \Rightarrow (\exists y')[y \xrightarrow{a^\mathfrak{N}} y' \text{ and } x' \preccurlyeq y']$
- Zag_a:** $x \preccurlyeq y$ and $y \xrightarrow{a^\mathfrak{N}} y' \Rightarrow (\exists x')[x \xrightarrow{a^\mathfrak{M}} x' \text{ and } x' \preccurlyeq y']$
- Up_p:** $x \preccurlyeq y$ and $x \in \|p\|^\mathfrak{M} \Rightarrow y \in \|p\|^\mathfrak{N}$
- Down_p:** $x \preccurlyeq y$ and $y \in \|p\|^\mathfrak{N} \Rightarrow x \in \|p\|^\mathfrak{M}$

By symmetry, the converse $\succsim: Y \rightsquigarrow X$ will also be a bisimulation between \mathfrak{N} and \mathfrak{M} .

The fundamental bisimulation-invariance property for sentences of the μ -calculus is the following.

Proposition 2.3 ([13], §5.3). *If \preccurlyeq is a bisimulation between \mathfrak{M} and \mathfrak{N} , then for all $x \in X$ and $y \in Y$, and all sentences $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$,*

$$x \preccurlyeq y \quad \Rightarrow \quad [x \in \|\varphi\|^\mathfrak{M} \Leftrightarrow y \in \|\varphi\|^\mathfrak{N}]$$

Proof. The conditions **Up_p** and **Down_p** give the base case of the induction, for atomic $p \in \Phi$, and the **Zig_a** and **Zag_a** conditions give the induction step for the $\langle a \rangle$ modalities. For μ -sentences, one uses the representation of $\|\mu Z.\varphi\|^\mathfrak{M}$ as a union of a chain of approximations and proceeds by transfinite induction. ■

When $\mathfrak{M} = \mathfrak{N}$ and $\preccurlyeq = \approx$ is also an equivalence relation on X , \approx is called a *bisimulation equivalence* on \mathfrak{M} . In this case, the (single-valued) quotient map $q: \mathfrak{M} \rightarrow \mathfrak{M}_\approx$ is a bisimulation between \mathfrak{M} and the quotient LTS \mathfrak{M}_\approx (well-defined, by the bisimulation conditions). It follows that for each sentence $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, the set $\|\varphi\|^\mathfrak{M}$ is a union of \approx equivalence classes. In particular, if \approx is a bisimulation equivalence of *finite* index, then for each fixed-point sentence $\mu Z.\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, the denotation $\|\mu Z.\varphi\|^\mathfrak{M}$ is a *finite* union of approximations $\|\varphi^n\|^\mathfrak{M}$, where $\varphi^0 \doteq \text{ff}$ and $\varphi^{n+1} \doteq \varphi[Z := \varphi^n]$ for $n < \omega$.

3 Bisimulations and Continuity

When written out so neatly, the zigzag conditions cry out to be analyzed as some variant on the theme of continuity. We observe a nice symmetry in subject and object: a preorder \preccurlyeq is a bisimulation, that is, *it respects* the structural components of an LTS model, exactly when the component transition relations and observation sets *respect it*, in the form of its topological structure as a preorder.

Recall that a preorder \preccurlyeq on a set is a reflexive and transitive binary relation. In the modal logic tradition, preorders give the relational Kripke semantics for **S4** modalities, with $\sigma(\preccurlyeq)$ interpreting \Diamond and $\tau(\preccurlyeq)$ interpreting \Box . For $A \subseteq X$, reflexivity gives $A \subseteq \sigma(\preccurlyeq)(A)$, and transitivity translates as

$\sigma(\preccurlyeq)\sigma(\preccurlyeq)(A) \subseteq \sigma(\preccurlyeq)(A)$; dually for $\tau(\preccurlyeq)$. The simplest topological structure associated with a preorder is its *Alexandroff topology*.

From work of McKinsey and Tarski in the 1940's, S4 also admits a more general topological semantics in addition to the (historically later) relational Kripke semantics using preorders. The axioms for \Box correspond to those of an arbitrary topological interior operator int_T , and dually, \Diamond corresponds to topological closure. Alexandroff topologies arise when one correlates the two semantics (see [3], where they go by the name D-topology, for "digital"). In earlier work on hybrid systems [11], Alexandroff spaces arising from *finite sub-topologies* of standard topologies on $X \subseteq \mathbb{R}^n$ (by the name "small" or AD-topologies) are used to model the conversion of sensor data into an input signal to a finite control automaton ([11], §5).

For binary relations or set-valued maps, the purely topological notion of continuity was introduced by Kuratowski and Bouligand in the 1930's, and generalizes that for single-valued functions (see, for example, [12], §4.4; that handbook article is a good source for a review of basic general topology.)

Definition 3.1 *Given a topological space (X, \mathcal{T}) , let $\mathcal{O}(\mathcal{T}) = \mathcal{T}$ and $\mathcal{C}(\mathcal{T})$ denote, respectively, the open and closed sets of \mathcal{T} .*

A relation $r : (X, \mathcal{T}) \rightsquigarrow (Y, \mathcal{S})$ is called:

- lower semi-continuous (l.s.c.) iff $U \in \mathcal{O}(\mathcal{S}) \Rightarrow \sigma(r)(U) \in \mathcal{O}(\mathcal{T})$
- upper semi-continuous (u.s.c.) iff $U \in \mathcal{O}(\mathcal{S}) \Rightarrow \tau(r)(U) \in \mathcal{O}(\mathcal{T})$
iff $C \in \mathcal{C}(\mathcal{S}) \Rightarrow \sigma(r)(C) \in \mathcal{C}(\mathcal{T})$
- continuous iff both l.s.c. and u.s.c.

Let $\text{Clop}(\mathcal{T}) = \mathcal{O}(\mathcal{T}) \cap \mathcal{C}(\mathcal{T})$ denote the Boolean algebra (under the finitary set-theoretic operations) of *clopen* subsets of (X, \mathcal{T}) . The two semi-continuity properties together imply that for every $A \in \text{Clop}(\mathcal{S})$, we have $\sigma(r)(A) \in \text{Clop}(\mathcal{T})$. In particular, the domain $\text{dom}(r) \doteq \sigma(r)(Y) \in \text{Clop}(\mathcal{T})$, since $Y \in \text{Clop}(\mathcal{S})$.

In the setting of recent work on hybrid control systems, where each of the components of an LTS model of a hybrid dynamical system is first-order definable in \mathbb{R}^n (and \mathbb{R}^{2n} for the transition relations) over an *o-minimal structure* $\bar{\mathbb{R}} = (\mathbb{R}; <, +, \cdot, 0, 1, \dots)$, the standard subspace topology on $X \subseteq \mathbb{R}^n$ is of obvious interest. The "tameness" of the topology in the o-minimal

setting ([6]) manifests itself in a *finite cell decomposition property*, and this is the core of a construction of a finite bisimulation equivalence in recent decidability results for hybrid systems [10]. Semi-continuity properties of the two sorts of transition relation of hybrid LTS models – evolution along functionally continuous semi-flows $\phi : X \times \mathbb{R}^+ \rightarrow X$, constrained within an invariant set, and reset relations $r : X \rightsquigarrow X$ modeling the effect of switching between discrete control modes – are discussed in [4] §4. In the typical case, each of the two sorts of transition relation will have as its domain a proper subset of X , which is closed but rarely also open. When viewed in metric terms, the upper semi-continuity property is particularly attractive.

Definition 3.2 *Given a relation $r : X \rightsquigarrow X$, we call a set $A \subseteq X$*

- up- r -closed iff $\sigma(\check{r})(A) \subseteq A$ iff $A \subseteq \tau(r)(A)$;
- down- r -closed iff $\sigma(r)(A) \subseteq A$ iff $A \subseteq \tau(\check{r})(A)$.

Let $Up(r), Dn(r) \subseteq \mathcal{P}(X)$ denote, respectively, the families of all up- r -closed and down- r -closed subsets of X .

In temporal logic or in the topological dynamics of set-valued functions, up- r -closed sets $A \subseteq X$ are also called *positive- or future-invariant* under r . When $r = \preceq$ is a preorder or partial order, it is usually written $\uparrow A = A$. Note that for arbitrary r , each of the families $Up(r)$ and $Dn(r)$ are closed under *both* arbitrary unions and arbitrary intersections, since the pre-image operators $\sigma(r)$ and $\tau(r)$ are completely additive and completely multiplicative respectively, and we can exploit the duality between r and \check{r} . Moreover, the two families are duals under complement: $A \in Up(r)$ iff $\neg A \in Dn(r)$. Thus the family of sets $UpDn(r) = Up(r) \cap Dn(r)$ is a complete Boolean algebra.

In the case of interest, where \preceq is a preorder, observe that: $A \in UpDn(\preceq)$ iff $\sigma(\succeq)\sigma(\preceq)(A) = A$ iff $\sigma(\preceq)(A) = A = \tau(\preceq)(A)$ iff A is a (disjoint) union of \preceq -clusters; that is, sets $C \subseteq X$ such that for all $x, y \in C$, $x \preceq y$ (all pairs of points in C are mutually \preceq -accessible).

Proposition 3.3 *Given an LTS $\mathfrak{M} = (X, \{a^\mathfrak{M}\}_{a \in \Sigma}, \{\|p\|^\mathfrak{M}\}_{p \in \Phi})$, and a pre-order \preceq on X , we have for each $a \in \Sigma$ and $p \in \Phi$, and all $A \in \mathcal{P}(X)$,*

$$\begin{aligned} \preceq \text{satisfies } \mathbf{Zig}_a &\quad \text{iff} \quad A \in Up(\preceq) \Rightarrow \sigma(a^\mathfrak{M})(A) \in Up(\preceq) \\ \preceq \text{satisfies } \mathbf{Zag}_a &\quad \text{iff} \quad A \in Dn(\preceq) \Rightarrow \sigma(a^\mathfrak{M})(A) \in Dn(\preceq) \\ \preceq \text{satisfies } \mathbf{Up}_p &\quad \text{iff} \quad \|p\|^\mathfrak{M} \in Up(\preceq) \\ \preceq \text{satisfies } \mathbf{Down}_p &\quad \text{iff} \quad \|p\|^\mathfrak{M} \in Dn(\preceq) \end{aligned}$$

Proof. The condition Zig_a for \preccurlyeq is equivalent to the inclusion: $\succcurlyeq \circ a^m \subseteq a^m \circ \succcurlyeq$, where \circ is relational composition, and this is in turn equivalent to:

$$\sigma(\succcurlyeq)\sigma(a^m)(A) \subseteq \sigma(a^m)\sigma(\succcurlyeq)(A)$$

for all $A \in \mathcal{P}(X)$. Then using the reflexivity of \preccurlyeq , so $A \in Up(\preccurlyeq)$ iff $A = \sigma(\succcurlyeq)(A)$, the stated equivalence follows. For the Zag_a condition, replace \succcurlyeq by \preccurlyeq . The equivalence for Up_p and Down_p are immediate from Definition 2.2. ■

Given a preorder \preccurlyeq on X , the *Alexandroff topology* $\mathcal{T}_{\preccurlyeq}$ determined by \preccurlyeq is the topology on X defined by simply taking $\mathcal{T}_{\preccurlyeq} = \mathcal{O}(\mathcal{T}_{\preccurlyeq}) = Up(\preccurlyeq)$ and $\mathcal{C}(\mathcal{T}_{\preccurlyeq}) = Dn(\preccurlyeq)$. Thus $\mathcal{T}_{\preccurlyeq}$ is closed under arbitrary intersections as well as unions, and for all $A \subseteq X$,

$$int_{\mathcal{T}_{\preccurlyeq}}(A) = \tau(\preccurlyeq)(A) \quad \text{and} \quad cl_{\mathcal{T}_{\preccurlyeq}}(A) = \sigma(\preccurlyeq)(A)$$

In particular, $Clop(\mathcal{T}_{\preccurlyeq}) = UpDn(\preccurlyeq)$ is a complete Boolean algebra. The topology $\mathcal{T}_{\preccurlyeq}$ has as a basis the collection of all sets $B_{\preccurlyeq}(x) \doteq \sigma(\succcurlyeq)(\{x\}) = \{y \in X \mid x \preccurlyeq y\}$, and $B_{\preccurlyeq}(x)$ is the intersection of all open sets in $\mathcal{T}_{\preccurlyeq}$ containing x .

More generally, a topology \mathcal{T} on X is called *Alexandroff* if it has the property that for every point $x \in X$, there is a *smallest open set* containing x . In particular, every *finite* topology on a (arbitrary) set X is Alexandroff. For a preorder \preccurlyeq on X , the topology $\mathcal{T}_{\preccurlyeq}$ is of course Alexandroff. Going the other way, any topology \mathcal{T} on X determines a relation $\preccurlyeq_{\mathcal{T}}$ on X , called the *specialization preorder* of \mathcal{T} , given by:

$$x \preccurlyeq_{\mathcal{T}} y \quad \text{iff} \quad (\forall U \in \mathcal{T})[x \in U \Rightarrow y \in U]$$

Note that $\preccurlyeq_{\mathcal{T}}$ is a *partial order* exactly when \mathcal{T} is T_0 , and is trivial (the identity relation) when \mathcal{T} is T_1 . Alexandroff topologies are those that can be completely recovered from their specialization preorder: for any preorder \preccurlyeq on X , $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$, and if \mathcal{T} is Alexandroff, then $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$. The Alexandroff topology on a preordered space can also be seen as a crude cousin of the Scott topology $\mathcal{T}_{\sqsubseteq}$ on a dcpo (X, \sqsubseteq) , which satisfies $\preccurlyeq_{\mathcal{T}_{\sqsubseteq}} = \sqsubseteq$; see [12], §2.4.

It follows immediately from Proposition 3.3 and Definition 3.1 that if (X, \mathcal{T}) is an Alexandroff space, then $a^m : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T})$ is l.s.c. with respect to \mathcal{T} iff $\preccurlyeq_{\mathcal{T}}$ satisfies Zig_a , and a^m is u.s.c. with respect to \mathcal{T} iff

$\preceq_{\mathcal{T}}$ satisfies **Zag_a**. The Alexandroff hypothesis is essential for this characterization of lower semi-continuity, but for *arbitrary* topological spaces (X, \mathcal{T}) , upper semi-continuity implies $\preceq_{\mathcal{T}}$ satisfies **Zag_a** (in longer words, $a^{\mathfrak{M}}$ is upper- $\preceq_{\mathcal{T}}$ -monotonic); see [12], §4.4.

We now have our topological characterization of bisimulation preorders.

Proposition 3.4 *Let $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p \in \Phi})$ be an LTS model and let \mathcal{T} be an Alexandroff topology on X . Then:*

$\preceq_{\mathcal{T}}$ is a bisimulation preorder on \mathfrak{M}
iff for each $a \in \Sigma$, $a^{\mathfrak{M}} : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T})$ is continuous, and
for each $p \in \Phi$, $\|p\|^{\mathfrak{M}} \in \text{Clop}(\mathcal{T})$

Moreover, the preorder

$$x \preceq_{\text{Clop}(\mathcal{T})} y \quad \text{iff} \quad (\forall A \in \text{Clop}(\mathcal{T})) [x \in A \Rightarrow y \in A]$$

includes $\preceq_{\mathcal{T}}$ and is symmetric, thus an equivalence relation $\approx_{\text{Clop}(\mathcal{T})}$. When $\preceq_{\mathcal{T}}$ is a bisimulation preorder on \mathfrak{M} , $\approx_{\text{Clop}(\mathcal{T})}$ is a bisimulation equivalence.

The last statement also follows from Proposition 3.3 and Definition 3.1, using the fact that $\text{Clop}(\mathcal{T}) = \text{Up}(\approx_{\text{Clop}(\mathcal{T})}) = \text{Dn}(\approx_{\text{Clop}(\mathcal{T})})$. Note that although $\preceq_{\mathcal{T}}$ and $\succ_{\mathcal{T}}$ are both bisimulations if either is such, the topological equivalence (Stone T₀ quotient) $\approx_{\mathcal{T}} = (\preceq_{\mathcal{T}} \cap \succ_{\mathcal{T}})$ can fail to be a bisimulation. If $B_{\mathcal{T}}(x) = B_{\preceq_{\mathcal{T}}}(x)$ and $C_{\mathcal{T}}(x) = \text{cl}_{\mathcal{T}}(\{x\})$ are, respectively, the smallest open and the smallest closed sets containing a point x , then under $\approx_{\mathcal{T}}$, the equivalence classes are $E_{\mathcal{T}}(x) = B_{\mathcal{T}}(x) \cap C_{\mathcal{T}}(x)$. In contrast, the equivalence class $D_{\text{Clop}(\mathcal{T})}(x)$, is the smallest clopen or $\preceq_{\mathcal{T}}$ -cluster containing both $B_{\mathcal{T}}(x)$ and $C_{\mathcal{T}}(x)$.

More generally, if \approx is any equivalence relation on X , and \mathcal{T}_{\approx} is the Alexandroff topology of \approx , then the basic open sets are just the equivalence classes under \approx , and $\mathcal{T}_{\approx} = \text{Clop}(\mathcal{T}_{\approx}) = \text{Up}(\approx) = \text{Dn}(\approx)$ is the complete Boolean algebra of all unions of equivalence classes. The *bisimulation equivalence* conditions **ZigZag_a** and **UpDn_p** reduce, respectively, to the requirement that the algebra $\text{UpDn}(\approx)$ be closed under $\sigma(a^{\mathfrak{M}})$, and that $\|p\|^{\mathfrak{M}} \in \text{UpDn}(\approx)$.

In the light of our excursion into general topology, we restate the basic truth-preservation property of bisimulations from Proposition 2.3.

Proposition 3.5 *Let $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p \in \Phi})$ be an LTS model and let \preccurlyeq be a bisimulation preorder on \mathfrak{M} .*

Then for every sentence $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$,

$$\sigma(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}}) = \|\varphi\|^{\mathfrak{M}} = \tau(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}})$$

hence $\|\varphi\|^{\mathfrak{M}} \in \text{Clop}(\mathcal{T}_{\preccurlyeq}) = \text{UpDn}(\preccurlyeq)$.

The truth-preservation property is: $\sigma(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}}) \subseteq \|\varphi\|^{\mathfrak{M}} \subseteq \tau(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}})$, and the reflexivity of \preccurlyeq gives the rest of the inclusions.

4 Algebraic semantics for the μ -calculus

For bisimulation preorders on \mathfrak{M} , the algebras of sets $\text{Clop}(\mathcal{T}_{\preccurlyeq})$ are of clearly of interest since they contain the denotations in \mathfrak{M} of all μ -calculus sentences. The algebraic perspective on the semantics of the μ -calculus is taken up in the recent work of Kwiatkowska and colleagues in [1], [2]. The enterprise in those papers is to extend the framework of Stone duality for Boolean algebras to modal algebras with fixed-points, and in the process, give an algebraic completeness proof for Kozen's axiomatization \mathbf{L}_μ of the μ -calculus, using a canonical model construction over the space of ultrafilters of the Lindenbaum algebra of the logic \mathbf{L}_μ . Their language for the μ -calculus contains logical constants **ff** and **tt**, but no alphabet Φ of propositional constants. We make the obvious extension.

Definition 4.1 *A structure $(\mathcal{A}, \{\sigma_a\}_{a \in \Sigma}, \{\|p\|^{\mathcal{A}}\}_{p \in \Phi})$ is called a modal algebra of signature (Φ, Σ) , with carrier \mathcal{A} , iff*

1. $(\mathcal{A}; \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra, with lattice order \leq ;
2. for each $p \in \Phi$, $\|p\|^{\mathcal{A}} \in \mathcal{A}$;
3. for each $a \in \Sigma$, $\sigma_a : \mathcal{A} \rightarrow \mathcal{A}$ is a finitely additive and normal operator with values in \mathcal{A} ,
i.e. for all $A, B \in \mathcal{A}$, $\sigma_a(A \vee B) = \sigma_a(A) \vee \sigma_a(B)$ and $\sigma_a(0) = 0$.

For modal formulas φ , the valuation $\|\varphi\|_\xi^A$ under a variable assignment $\xi : \text{PVar} \rightarrow \mathcal{A}$ is defined inductively, analogous to Definition 2.1.

Such a structure is called a modal μ -algebra if for each formula $\mu Z. \varphi \in \mathcal{F}_\mu(\Phi, \Sigma)$, the \leq -monotone operator $A \mapsto \|\varphi\|_{\xi(A/Z)}^A$ has a least pre-fixed-point in \mathcal{A} , in which case:

$$\begin{aligned}\|\mu Z. \varphi\|_\xi^A &\stackrel{\text{def}}{=} \bigwedge \{A \in \mathcal{A} \mid \|\varphi\|_{\xi(A/Z)}^A \leq A\} \\ &= \bigwedge \{A \in \mathcal{A} \mid \|\varphi\|_{\xi(A/Z)}^A = A\}\end{aligned}$$

Definition 4.2 A modal frame of signature (Φ, Σ) is a pair $(\mathfrak{M}, \mathcal{A})$ where $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p \in \Phi})$ is an LTS model and $\mathcal{A} \subseteq \mathcal{P}(X)$ is a (set-theoretic) modal algebra for \mathfrak{M} ; that is, \mathcal{A} is a Boolean algebra under the finitary set-theoretic operations, contains each of the sets $\|p\|^{\mathfrak{M}}$ and is closed under each of the pre-image operators $\sigma(a^{\mathfrak{M}})$.

A modal μ -frame is a modal frame $(\mathfrak{M}, \mathcal{A})$ such that \mathcal{A} is a modal μ -algebra. An LTS \mathfrak{M} can be identified with the modal μ -frame $(\mathfrak{M}, \mathcal{P}(X))$.

For purely modal formulas $\varphi \in \mathcal{F}(\Phi, \Sigma)$, the semantics in $(\mathfrak{M}, \mathcal{P}(X))$ and in any modal frame $(\mathfrak{M}, \mathcal{A})$ are in agreement: $\|\varphi\|_\xi^{\mathfrak{M}} = \|\varphi\|_\xi^A$ for all variable assignments $\xi : \text{PVar} \rightarrow \mathcal{A}$. But in general, they part company on μ -formulas, since the smallest set in \mathcal{A} such that some condition holds will in general be larger than the smallest of all subsets of X such that the same condition holds. This motivates the following definition.

Definition 4.3 Given an LTS model \mathfrak{M} and a modal μ -algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ for \mathfrak{M} , we say the frame $(\mathfrak{M}, \mathcal{A})$ is in semantic agreement with \mathfrak{M} iff for all formulas $\varphi \in \mathcal{F}_\mu(\Phi, \Sigma)$ and all assignments ξ in \mathcal{A} , we have: $\|\varphi\|_\xi^A = \|\varphi\|_\xi^{\mathfrak{M}}$.

In other words, such algebras \mathcal{A} yield the “true” denotation of formulas, relative to the standard set-theoretic semantics in \mathfrak{M} . In establishing semantic agreement, the point is to show that for assignments ξ in \mathcal{A} , each set $\|\mu Z. \varphi\|_\xi^{\mathfrak{M}}$ is in \mathcal{A} ; the fact that $\|\mu Z. \varphi\|_\xi^{\mathfrak{M}}$ is then the least pre-fixed-point of $A \mapsto \|\varphi\|_{\xi(A/Z)}^A = \|\varphi\|_{\xi(A/Z)}^{\mathfrak{M}}$ follows by induction.

In [1] and [2], modal algebras $\mathcal{A} \subseteq \mathcal{P}(X)$ are thought of as providing a clopen basis for a topology $\mathcal{T}_\mathcal{A}$ on X , inspired by Stone duality. They concentrate on algebras \mathcal{A} which are *perfect* and *reduced* as fields of sets, since those conditions characterize $(X, \mathcal{T}_\mathcal{A})$ being a *Stone space* – that is, compact, Hausdorff and totally disconnected. In that case, Stone duality gives $X \cong$

$Ult(\mathcal{A})$, where $Ult(\mathcal{A})$ is the space of ultrafilters of \mathcal{A} . They further specialize to *descriptive* modal frames $(\mathfrak{M}, \mathcal{A})$, which have the additional property that for each $a \in \Sigma$, the relation $a^{\mathfrak{M}} : X \rightsquigarrow X$ can be recovered from the algebra, in the sense that $x \xrightarrow{a^{\mathfrak{M}}} y$ iff $(\forall A \in \mathcal{A})[x \in \tau(a^{\mathfrak{M}})(A) \Rightarrow y \in A]$. In [2] §6, it is established that if $(\mathfrak{M}, \mathcal{A})$ is a descriptive modal μ -frame, then $(\mathfrak{M}, \mathcal{A})$ is in semantic agreement with \mathfrak{M} .

Our analysis of bisimulation preorders leads to an alternative and simpler condition for semantic agreement.

Proposition 4.4 *If \preccurlyeq is a bisimulation preorder on an LTS model \mathfrak{M} , then $(\mathfrak{M}, Clop(\mathcal{T}_{\preccurlyeq}))$ is in semantic agreement with \mathfrak{M} .*

Proof. From Proposition 3.4, $Clop(\mathcal{T}_{\preccurlyeq})$ is a modal algebra for \mathfrak{M} , since it contains each $\|p\|^{\mathfrak{M}}$ and is closed under $\sigma(a^{\mathfrak{M}})$. The completeness of $Clop(\mathcal{T}_{\preccurlyeq})$ as a Boolean algebra ensures that it is also a μ -algebra, since the relevant pre-fixed-points exist in $Clop(\mathcal{T}_{\preccurlyeq})$. From Proposition 3.5, for all sentences $\varphi \in \mathcal{S}_{\mu}(\Phi, \Sigma)$, we have $\|\varphi\|^{\mathfrak{M}} \in Clop(\mathcal{T}_{\preccurlyeq})$. To prove that $\|\varphi\|_{\xi}^{\mathfrak{M}} \in Clop(\mathcal{T}_{\preccurlyeq})$ for all formulas $\varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$ and all assignments ξ in $Clop(\mathcal{T}_{\preccurlyeq})$, one can proceed directly by induction on complexity of formulas, exploiting the representation of $\|\mu Z. \varphi\|_{\xi}^{\mathfrak{M}}$ as a union of a chain of approximations, as one does in the proof of Proposition 2.3. ■

The family of standard denotations of sentences gives us the *simplest* modal algebra in semantic agreement. Define

$$\mathcal{S}_{\mu}^{\mathfrak{M}} \doteq \{\|\varphi\|^{\mathfrak{M}} \mid \varphi \in \mathcal{S}_{\mu}(\Phi, \Sigma)\}$$

Then $\mathcal{S}_{\mu}^{\mathfrak{M}}$ is clearly a modal μ -algebra: an assignment ξ in $\mathcal{S}_{\mu}^{\mathfrak{M}}$ maps variables V_i to sets $\|\psi_i\|^{\mathfrak{M}}$, so for any formula $\mu Z. \varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$, we have $\|\mu Z. \varphi\|_{\xi}^{\mathfrak{M}} = \|\mu Z. \varphi[V_i := \psi_i]\|^{\mathfrak{M}} \in \mathcal{S}_{\mu}^{\mathfrak{M}}$. Moreover, $\mathcal{S}_{\mu}^{\mathfrak{M}}$ is the *smallest* modal μ -algebra \mathcal{A} such that $(\mathfrak{M}, \mathcal{A})$ is in semantic agreement with \mathfrak{M} .

The quest for a bisimulation equivalence of finite index on an LTS model \mathfrak{M} is often represented (e.g. [7] §3.1; [10] §2) as an algorithm which starts with the coarsest partition of the state space that respects the sets $\|p\|^{\mathfrak{M}}$, so satisfies the conditions $UpDn_p$, then constructs successively finer partitions according to the $ZigZag_a$ conditions. Algebraically, this amounts to generating of a sequence of Boolean algebras $\mathcal{S}_k^{\mathfrak{M}}$ for $k < \omega$, where

$$\mathcal{S}_k^{\mathfrak{M}} \doteq \{\|\varphi\|^{\mathfrak{M}} \mid \varphi \in \mathcal{S}^k(\Phi, \Sigma)\}$$

is the finite Boolean algebra of denotations of *modal* sentences of modal degree $\leq k$. So $\mathcal{S}_{k+1}^{\mathfrak{M}}$ is the smallest Boolean algebra generated by $\mathcal{S}_k^{\mathfrak{M}} \cup \{\sigma(a^{\mathfrak{M}})(A) \mid A \in \mathcal{S}_k^{\mathfrak{M}}\}$. The algorithm terminates at stage $k+1$ if $\mathcal{S}_{k+1}^{\mathfrak{M}} = \mathcal{S}_k^{\mathfrak{M}}$, in which case the equivalence relation:

$$x \approx_{\mathcal{S}_k^{\mathfrak{M}}} y \quad \doteq \quad (\forall A \in \mathcal{S}_k^{\mathfrak{M}}) [x \in A \Leftrightarrow y \in A]$$

is a finite bisimulation equivalence whose equivalence classes are atoms of the algebra $\mathcal{S}_k^{\mathfrak{M}}$, and $\mathcal{S}_{\mu}^{\mathfrak{M}} = \mathcal{S}_k^{\mathfrak{M}}$.

5 Discussion

Our larger interest is in polymodal extensions of the modal μ -calculus as a broad logical framework for the formal analysis of hybrid control systems. This theme is developed in a companion paper [4]. The idea is to take the basic LTS model \mathfrak{M} as a skeleton, and “flesh it out” by imbuing the state space with topological, metric tolerance or other structure, and extending the μ -calculus accordingly. Using the modal logic **S4**, we can represent the *real* topology on X as a subspace of \mathbb{R}^n , and express continuity properties of the component transition relations. We can also represent a bisimulation preorder \preccurlyeq on \mathfrak{M} , axiomatizing the basic bisimilarity conditions by:

$$\begin{aligned} \mathbf{Zig}_a = \mathbf{l.s.c.}_a : & \quad \langle a \rangle \Box Z \rightarrow \Box \langle a \rangle Z \\ \mathbf{Zag}_a = \mathbf{u.s.c.}_a : & \quad \Diamond \langle a \rangle Z \rightarrow \langle a \rangle \Diamond Z \\ \mathbf{Up}_p = \mathbf{open}_p : & \quad p \rightarrow \Box p \\ \mathbf{Dn}_p = \mathbf{closed}_p : & \quad \Diamond p \rightarrow p \end{aligned}$$

If in addition, we also want the preorder \preccurlyeq to preserve the truth and denotation of sentences in the expanded language $\mathcal{S}_{\mu, \Box}(\Phi, \Sigma)$, then we have to add an extra pair of structure-preservation clauses. For preorders, $\mathbf{Zig}_{\preccurlyeq}$ translates as the condition of weak-directness, given by the scheme $\Diamond \Box Z \rightarrow \Box \Diamond Z$, and $\mathbf{Zag}_{\preccurlyeq}$ becomes the trivial $\Diamond \Diamond Z \rightarrow \Diamond \Diamond Z$. The modal logic **KTB** is that of reflexive and symmetric relations, via which we can represent the metric tolerance relation of differing by distance ϵ , for particular $\epsilon > 0$. As the logic of equivalence relations, **S5** is of obvious interest.

For polymodal extensions of the μ -calculus, algebraic semantics and Stone duality theory offer an available means to completeness of the proof systems

[5]. As noted in [1], a deficiency of Walukiewicz' direct proof of completeness of L_μ (see [14]) with respect to the standard set-theoretic semantics is that it does not lift to extensions of the logic.

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